PHYS 798C Spring 2024
Lecture 25 Summary

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I. DC SQUIDS

A SQUID is a Superconducting QUantum Interference Device. The dc SQUID acts as a single Josephson junction with a flux-tunable critical current. The renormalized critical current is $\widetilde{I}_c(\Phi) = 2I_c \cos\left(\pi \frac{\Phi}{\Phi_0}\right)$, the renormalized phase is given by $\widetilde{\gamma} = \gamma_1 + \pi\frac{\Phi}{\Phi_0}$, the renormalized resistance is $\widetilde{R} = R/2$, and the renormalized capacitance is given by $\widetilde{C} = 2C$.
(Note that in this derivation we assume that the "self-flux" produced by screening currents in the loop is small, or in other words $LI_c << \Phi_0$, where L is the self-inductance of the loop.)

The critical current of the SQUID is a periodic function of flux, repeating every time Φ advances through Φ_0. It ranges in value from $2I_c$ to zero, periodically in flux. Consider the case of a SQUID with small capacitance. It will have an I-V curve given by,

$$\langle V \rangle = \begin{cases} 0 & I < \widetilde{I}_c \\ \widetilde{R}_N \sqrt{I^2 - \widetilde{I}_c^2(\Phi)} & I > \widetilde{I}_c \end{cases}$$

where $\widetilde{I}_c$ can be modulated between $2I_c$ and 0, depending on the flux applied to the SQUID. If we now bias the SQUID with a current just under $2I_c$, the voltage developed on the SQUID will be a function of flux applied. The dependence will be periodic in flux with period Φ_0, but not sinusoidal. The transfer function between voltage and flux is nonlinear, but can be linearized for small ranges of applied flux.

As an example, consider a SQUID with loop area $A = 1$ cm^2. Assume that a flux-locked loop (FLL) can be prepared so that the sensitivity of the SQUID is 10^{-3} Φ_0. The resolution in magnetic field is $\delta B = \delta\Phi/A = 20$ fT. This should be compared to the earth's magnetic field, which is roughly 50 μT. A human brain produces magnetic fields on the order of 200 fT outside the head when thoughts are taking place...

A flux-locked loop is created by taking the voltage drop on the SQUID and converting it partly in to a feedback current (through resistor R) that runs through an inductor that is coupled to the SQUID through a mutual inductance M. The sense of the coupling loop is chosen so that the feedback current creates a flux that opposes the change in flux coming from external fields. This keeps the SQUID exposed to a fixed amount of flux, so that it operates near the region of linear flux-to-voltage transduction. The "error signal" voltage in the feedback then gives the change in flux to the SQUID as $\Delta\Phi_{applied} = \frac{M}{R}\Delta V_{out}$. A typical FLL circuit will have a noise level of $10^{-6}\Phi_0/\sqrt{Hz}$.

dc SQUIDs find many uses. The class web site gives examples of underground wireless radio, magnetoencephalography, and π SQUIDs arising from superconductors with non-s-wave order parameter.

II. FLUCTUATIONS IN SUPERCONDUCTORS

So far, everything we have discussed has been in "mean field theory", which means that fluctuations have been ignored. By fluctuations we mean that there are microscopic processes in which the superconductor borrows energy $k_B T$ from the thermal bath and explores states other than those described by the equilibrium order parameter. These fluctuations are dynamic and occurring many times per second in small regions all over the material. Our measurements typically integrate over many many such fluctuations, so we see a time-averaged result of all these fluctuations. We will explore the spatial and temporal aspects of the fluctuations.
Clear manifestations of fluctuations are seen in lower dimensional samples. We will consider 0D, 1D, 2D and 3D samples. The dimensionality depends on how big the dimensions of the sample are compared to the GL coherence length ξ_{GL}.
Measurements of resistance vs. temperature show a decrease of resistance above T_c, often as a "rounding" of the transition $R(T)$. In addition we saw earlier (Lecture 16) that a 1D superconductor can suffer thermally-induced phase-slip events that extend the resistive tail below T_c. This was a case in which the

superconductor can borrow energy k_BT from the thermal bath to locally extinguish superconductivity ($|\psi| \to 0$ at some point) and allow a phase-slip event to occur.
Another experimental manifestation of fluctuations is magnetic susceptibility, which shows diamagnetic response above T_c, particularly in lower-dimensional samples. This results from superconducting fluctuations which help to create "evanescent" superconductivity and diamagnetism above T_c.

The effects of fluctuations turn out to be strongest near T_c. They can be modeled very well by Ginzburg-Landau theory using the free energy expansion, especially since GL theory is ideally suited for calculations near T_c.

III. QUALITATIVE EFFECTS OF FLUCTUATIONS IN SUPERCONDUCTORS

Consider the GL free energy density difference expansion in the order parameter:
$f_s = f_n + \alpha\,|\psi|^2 + \frac{\beta}{2}\,|\psi|^4 + \frac{1}{2m^*}\left|\left(\frac{\hbar}{i}\vec{\nabla} - e^*\vec{A}\right)\psi\right|^2 + \frac{\mu_0 h^2}{2}$.
We will consider a homogeneous system with no external fields, and just consider,
$f_s - f_n = \alpha\,|\psi|^2 + \frac{\beta}{2}\,|\psi|^4$.
1) Above T_c we have $\alpha > 0$ and the equilibrium value of $\langle\psi\rangle = 0$. However the system can borrow k_BT of energy and explore other states where at least temporarily $\langle\psi\rangle \neq 0$. As a result we expect $\langle\psi^2\rangle \neq 0$.
2) Below T_c we have $\alpha < 0$ and the equilibrium value of $\langle\psi\rangle \neq 0$. Again the system can explore other states beyond the equilibrium mean-field solution. The question is this: what measurable effects do these fluctuations have?

IV. QUANTITATIVE EFFECTS OF FLUCTUATIONS IN 0D SUPERCONDUCTORS

A zero-dimensional (0D) superconductor is a nano-particle that has characteristic dimension R which is much smaller than the GL coherence length ξ_{GL} and the effective screening length λ_{eff}. As such it cannot support any gradients in the order parameter or any screening. It is a superconducting dot that just "varies in intensity" as a function of time. The free energy difference is simply given by,
$F_s - F_n = \int (f_s - f_n) d^3r = V(\alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4)$, where V is the volume of the particle.
As calculated before, in the superconducting state ($T < T_c$ and $\alpha < 0$), we know the equilibrium value of the order parameter is $|\psi_0|^2 = -\alpha/\beta = \frac{\alpha_0(1-t)}{\beta}$, and the mean field free energy is $F_0 = -\frac{\alpha^2}{2\beta}V = -\frac{\alpha_0^2(1-t)^2}{2\beta}V = -\frac{\mu_0 H_c^2}{2}V$. In the last version we use the definition of the condensation energy density.

Now assume that due to a fluctuation, $\psi = \psi_0 + \delta\psi$, and do an expansion of the free energy difference F to second order in $\delta\psi$. One finds,
$F - F_0 = \frac{1}{2}\left(\frac{\partial^2 F}{\partial|\psi|^2}\right)_{\psi_0}(\delta\psi)^2$.
Using the GL free energy above, we find,
$F - F_0 = \frac{1}{2}(-4\alpha V)(\delta\psi)^2$.

We now make the assumption of energy equipartition for a 0D system in equilibrium with a thermal bath at temperature T, and assign the single degree of freedom of this system to have on average k_BT of energy;
$\langle F - F_0\rangle = 2\alpha_0(1-t)V\langle(\delta\psi)^2\rangle = k_BT$.

With this, the normalized fluctuation can be written as,
$\frac{\langle(\delta\psi)^2\rangle}{|\psi_0|^2} = \frac{k_BT}{V}\frac{\beta}{2\alpha_0^2(1-t)^2}$, again assuming $t < 1$. This result says that fluctuations will be large when we consider 0D particles with small volume V, high-T_c superconductors, and most-importantly, temperatures close to T_c.
Note that this treatment assumes small fluctuations, so that the ratio $\frac{\langle(\delta\psi)^2\rangle}{|\psi_0|^2} << 1$. When this ratio grows to be on the order of unity, one has "critical fluctuations." In other words, when the fluctuations are on the order of the mean value, one should take a very different approach to calculating the effects on physical quantities. This is the study of "critical phenomena."

One can also write this result as,
$4(\frac{\mu_0 H_c^2}{2}V)\frac{\langle(\delta\psi)^2\rangle}{|\psi_0|^2} = k_B T$. The left hand side is the product of the condensation energy of the particle and the typical normalized fluctuation size, and it is equated to the thermal bath energy. This argues that if either the condensation energy or volume are "large", the fluctuations will be small.

For the case of $T > T_c$ the result for a typical fluctuation is,
$\langle(\delta\psi)^2\rangle = \frac{k_B T}{\alpha_0(t-1)V}$, which differs from the result below T_c by a factor of 2.
These results suggest that the fluctuations diverge at T_c, but that is due to the approximation of ignoring the higher order terms in the free energy expansion, which keep the fluctuations finite.
A more careful treatment calculates the full order parameter (mean field plus fluctuations) as,
$\langle|\psi|^2\rangle = \int |\psi|^2 e^{-F/k_B T} d^2\psi / \int e^{-F/k_B T} d^2\psi$, where the integral is over the complex order parameter, and the free energy includes the quartic term. In this case the order parameter remains finite through T_c, where it has a value,
$\langle(\delta\psi)^2\rangle_{T_c} \cong \sqrt{\frac{2k_B T_c}{V\beta}}$. This result also suggests that 0D systems with high T_c and small volume will show the strongest fluctuations.

Note that the strength of fluctuations grows like $\sqrt{T_c}$. Hence room temperautre (or higher) superconductors are going to experinece very strong fluctuaitons near the transition temperature. Among other things, this will create a significant broadening of the resistive transition, particularly in a magnetic field. The lack of broadening of these transitions has been cited as a reason to question data that claims to demonstrate near-room temperature superconductors.

Fluctuations are evident in measurements of the magnetic susceptibility of small (0D) superconducting grains. The magnetic susceptibility is the ratio of the magnetization developed in a sample to the applied magnetic field, $\chi = M/H$. In the Meissner state the superconductor will develop a negative susceptibility. Due to fluctuations, there will be a non-zero susceptibility at and above the bulk transition temperature. The magnetic susceptibility is calculated to be,
$\chi = -\frac{1}{40\pi}\frac{\mu_0 R^2 |\psi|^2 (e^*)^2}{m^*}$,
where R is the radius of the grain. Plots of magnetic susceptibility vs. temperature for Al powders with three different grain sizes are shown on the class web site. All show substantial non-zero values for $\langle|\psi|^2\rangle$ up to 10% above T_c.